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2008 J. Phys. A: Math. Theor. 41 365207

(http://iopscience.iop.org/1751-8121/41/36/365207)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 365207 (16pp) doi:10.1088/1751-8113/41/36/365207

Casimir operators induced by the Maurer–Cartan equations

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Received 20 May 2008, in final form 14 July 2008
Published 1 August 2008
Online at stacks.iop.org/JPhysA/41/365207

Abstract

It is shown that for inhomogeneous Lie algebras with only one Casimir operator, the latter can be explicitly constructed from the Maurer–Cartan equations by means of wedge products. It is further proved that this constraint imposes sharp bounds for the dimension of the representation R defining the semidirect product. The procedure is generalized to compute also the rational invariant of some Lie algebras.

PACS number: 02.20.Sv

1. Introduction

The study of (generalized) Casimir operators of symmetry groups has acquired an important fundamental for the understanding of physical theories, where they constitute a valuable tool for classification schemes and the subsequent establishment of mass formulae. The invariants of a Lie algebra serve principally to label irreducible representations and are therefore of interest to separate multiplets, their eigenvalues being identified with the quantum numbers of certain physical observables. Among the different physical situations where Lie algebras appear, like in geometric symmetry groups related to degeneracies of energy levels or spectrum generating algebras associated with non-invariance groups, the most important case probably corresponds to those symmetries depending on the potential energy functions (e.g., Coulomb problem), nowadays known as dynamical symmetry groups. For these the invariants are specially useful, since they enable us to express the Hamiltonian of the system in terms of the Casimir operators of a Lie algebra and some of its Lie subalgebras (e.g., the interaction boson model) [1–3]. Other situations involving the invariants of an algebra and certain of its subalgebras appear in connection with symmetry breaking problems [4].

The question how to find the invariants of a Lie algebra has been approached by many different methods, from the study of enveloping algebras of semisimple Lie algebras to analytical methods based on the integration of a set of linear first-order partial differential equations [5–13]. Other methods, such as tensor operator techniques or algebraic reductions, have been successfully developed for special classes of Lie algebras [14–16]. Another possibility of obtaining the Casimir operators, or more generally, the invariants by the coadjoint representation, is the use of the structural or Maurer–Cartan equations of the Lie algebra. Differential forms, being a powerful tool for many physical problems, have however not been used extensively in the invariant problem, although they are a common technique in the study of Poisson structures and related integrability conditions [2, 17, 18]. This interpretation is very close to that of total differential equations [11] and suggests that for Lie algebras with one Casimir operator, the latter could be somehow encoded in the Maurer–Cartan equations, by means of wedge products. On the other hand, algebras with only one invariant cannot be obtained by contraction of semisimple Lie algebras (up to the trivial case in dimension three [16]), thus the invariant has to be computed directly.

In this work we prove that for semidirect products of semisimple and Abelian Lie algebras with only one invariant for the coadjoint representation, the Casimir operator of the algebra is completely determined by the corresponding Maurer–Cartan equations. As a consequence, such algebras will be endowed with a concrete geometrical structure. We also point out that the procedure enables us to find the commuting polynomials for the cases where the Lie algebras have a rational invariant, and even allows us to recover the invariants for Lie algebras with two Casimir operators. These results constitute a complement to the geometrical method developed recently in [15, 19] for Lie algebras with certain structure. As applications, we derive a basis-independent expression for the Casimir operator of the special affine Lie algebras $\mathfrak{sa}(n\mathbb{R})$ used in affine quantum gauge theories [2].

Any Lie algebra $\mathfrak{g}$ and any representation R considered in this work are defined over the field $\mathbb{R}$ of real numbers. We convene that non-written brackets are either zero or obtained by antisymmetry. We also assume implicitly the Einstein summation convention. Abelian Lie algebras of dimension n will be denoted by nL_1.

2. Invariants of Lie algebras: Maurer–Cartan equations

Among the multiple algebraic and analytical methods developed in the literature in order to determine the Casimir invariants of Lie algebras, the procedure based on partial differential equations (PDEs) has probably become the most standard [9, 11]. Given a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}$ and the structure tensor $\left\{C_{ij}^k\right\}$, we consider the realization of $\mathfrak{g}$ in the space $C^\infty(\mathfrak{g}^*)$ determined by the differential operators:

$$\widehat{X}_i = C_{ij}^k x_k \frac{\partial}{\partial x_j}, \tag{1}$$

where $[X_i, X_j] = C_{ij}^k X_k (1 \leqslant i < j \leqslant n, 1 \leqslant k \leqslant n)$. It is straightforward to verify that the brackets $[\widehat{X}_i, \widehat{X}_j] = -C_{ij}^k \widehat{X}_k$ are satisfied, showing that they define a linear representation of the algebra. The Casimir operators of $\mathfrak{g}$ correspond to functions on the generators $F(X_1, \ldots, X_n)$ of $\mathfrak{g}$ that satisfy the constraint

$$[X_i, F(X_1, \ldots, X_n)] = 0. \tag{2}$$

Using the ansatz by PDEs, it can be seen that Casimir operators constitute a special case corresponding to polynomial solutions to the system of linear first-order partial differential equations:

$$\widehat{X}_i F(x_1, \ldots, x_n) = C_{ij}^k x_k \frac{\partial F}{\partial x_j}(x_1, \ldots, x_n) = 0, \qquad 1 \leqslant i \leqslant n. \tag{3}$$

Solutions of (3) are called generalized Casimir invariants or invariants of the coadjoint representation of $\mathfrak{g}$. The classical Casimir operators are recovered using the symmetrization map $\mathrm{Sym}\left(x_1^{a_1}\cdots x_p^{a_p}\right) = \frac{1}{k!}\sum_{\sigma\in S_p} x_{\sigma(1)}^{a_1}\cdots x_{\sigma(p)}^{a_p}$ and then replacing the variables x_i by the corresponding generator X_i [8]. A maximal set of functionally independent solutions of (3) will be called a fundamental set of invariants. The number $\mathcal{N}(\mathfrak{g})$ of independent solutions to the system is given by [5]

$$\mathcal{N}(\mathfrak{g}) = \dim\mathfrak{g} - \sup_{x_1,\ldots,x_n}\mathrm{rank}\left(C_{ij}^k x_k\right), \tag{4}$$

where $A(\mathfrak{g}) := \left(C_{ij}^k x_k\right)$ is the matrix which represents the commutator table over the basis $\{X_1,\ldots,X_n\}$. Some years ago, a method based on total differential equations was proposed in [11]. For algebras with one invariant, system (3) is simplified to a total differential equation of the type

$$\mathrm{d}F = \mathrm{d}x_1 + U_{12}\,\mathrm{d}x_2 + \cdots + U_{1n}\,\mathrm{d}x_n = 0, \tag{5}$$

by means of successive reductions, where U_{1i} are functions of the generators of $\mathfrak{g}$ obtained from the commutator matrix $A(\mathfrak{g})$ [11]. The solution of (5) is thus a first integral $F = \sum_{i=1}^n f_i x_i$ of equation (5), with f_i being the result of deleting the common factors in U_i. This method reduced the computation of the Casimir operator to the evaluation of $\dim(\mathfrak{g}) - 1$ determinants and the integration of the total differential equation (5). This ansatz can be easily reformulated in terms of differential forms, which constitute a more natural frame for this type of algebras. In terms of the Maurer–Cartan equations, the Lie algebra $\mathfrak{g}$ is described as follows: given the structure tensor $\left\{C_{ij}^k\right\}$ over the basis $\{X_1,\ldots,X_n\}$, the identification of the dual space $\mathfrak{g}^*$ with the left-invariant Pfaffian forms on the simply connected Lie group whose algebra is isomorphic to $\mathfrak{g}$ allows us to define an exterior differential d on $\mathfrak{g}^*$ by

$$\mathrm{d}\omega(X_i, X_j) = -C_{ij}^k\omega(X_k), \qquad \omega\in\mathfrak{g}^*. \tag{6}$$

With this operator co boundary d we can rewrite $\mathfrak{g}$ as a closed system of 2-forms,

$$\mathrm{d}\omega_k = -C_{ij}^k\omega_i\wedge\omega_j, \qquad 1\leqslant i<j\leqslant\dim(\mathfrak{g}), \tag{7}$$

called the Maurer–Cartan equations of $\mathfrak{g}$. In particular, the condition $\mathrm{d}^2\omega_i = 0$ for all i is equivalent to the Jacobi condition. In order to relate this approach with the number of invariants, we consider the linear subspace $\mathcal{L}(\mathfrak{g}) = \mathbb{R}\{\mathrm{d}\omega_i\}_{1\leqslant i\leqslant\dim\mathfrak{g}}$ of $\bigwedge^2\mathfrak{g}^*$ generated by the 2-forms $\mathrm{d}\omega_i$ [20]. It follows at once that $\dim\mathcal{L}(\mathfrak{g}) = \dim(\mathfrak{g})$ if and only if $\mathrm{d}\omega_i\neq 0$ for all i, that is, if the condition $\dim(\mathfrak{g}) = \dim[\mathfrak{g},\mathfrak{g}]$ holds. If $\omega = a^i\,\mathrm{d}\omega_i (a^i\in\mathbb{R})$ is a generic element of $\mathcal{L}(\mathfrak{g})$, then we can find $j_0(\omega)\in\mathbb{N}$ such that

$$\bigwedge^{j_0(\omega)}\omega\neq 0, \qquad \bigwedge^{j_0(\omega)+1}\omega\equiv 0. \tag{8}$$

The quantity $j_0(\mathfrak{g})$ defined by

$$j_0(\mathfrak{g}) = \max\{j_0(\omega)|\omega\in\mathcal{L}(\mathfrak{g})\} \tag{9}$$

constitutes a numerical invariant of the Lie algebra $\mathfrak{g}$, and allows us to rewrite equation (4) as

$$\mathcal{N}(\mathfrak{g}) = \dim\mathfrak{g} - 2j_0(\mathfrak{g}). \tag{10}$$

This identity implies that $j_0(\mathfrak{g})$ coincides with the number of internal labels necessary to describe a general irreducible representation of $\mathfrak{g}$ [20–22].

This reformulation in terms of differential forms can be useful to construct the Casimir operator when the condition $\mathcal{N}(\mathfrak{g}) = 1$ is satisfied. As an example how it can be computed from the Maurer–Cartan equations, consider the five-dimensional Lie algebra $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})\overrightarrow{\oplus}_{D_{\frac{1}{2}}} 2L_1$ given by the brackets

$$[X_1, X_2] = 2X_2, \quad [X_1, X_3] = -2X_3, \quad [X_2, X_3] = X_1, \quad [X_1, X_4] = X_4,$$
$$[X_1, X_5] = -X_5, \quad [X_2, X_5] = X_4, \quad [X_3, X_4] = X_5.$$

The structure equations are easily seen to be

$$\begin{aligned} &\mathrm{d}\omega_1 = -\omega_2 \wedge \omega_3, && \mathrm{d}\omega_2 = -2\omega_1 \wedge \omega_2, && \mathrm{d}\omega_3 = 2\omega_1 \wedge \omega_3,\\ &\mathrm{d}\omega_4 = -\omega_1 \wedge \omega_4 - \omega_2 \wedge \omega_5, && \mathrm{d}\omega_5 = \omega_1 \wedge \omega_5 - \omega_3 \wedge \omega_4. \end{aligned} \tag{11}$$

Let us consider a generic element $\omega \in \mathfrak{g}^*$: $\omega = a^1\omega_1 + a^2\omega_2 + a^3\omega_3 + a^4\omega_4 + a^5\omega_5$, where $a^i \in \mathbb{R}$ are considered as variables. The coboundary operator d is given by

$$\begin{aligned} \mathrm{d}\omega = {} & a^1\omega_2 \wedge \omega_3 + 2a^2\omega_1 \wedge \omega_2 - 2a^3\omega_1 \wedge \omega_3 + a^4(\omega_1 \wedge \omega_4 + \omega_2 \wedge \omega_5)\\ & + a^5(-\omega_1 \wedge \omega_5 + \omega_3 \wedge \omega_4). \end{aligned}$$

Now, computing the wedge product $\omega \wedge \mathrm{d}\omega \wedge \mathrm{d}\omega$ and expanding it, we obtain the expression

$$\omega \wedge \mathrm{d}\omega \wedge \mathrm{d}\omega = -6(a^1a^4a^5 + a^2(a^5)^2 - a^3(a^4)^2)\omega_1 \wedge \omega_2 \wedge \omega_3 \wedge \omega_4 \wedge \omega_5. \tag{12}$$

Observe that the polynomial $\Phi = a^1a^4a^5 + a^2(a^5)^2 - a^3(a^4)^2$ is homogeneous of degree 2 in the variables a^4 and a^5, corresponding to the generators in the radical of $\mathfrak{g}$. Now, replacing the variables a^i by the corresponding x_i, it is straightforward to verify that $C = x_1x_4x_5 + x_2x_5^2 - x_3x_4^2$ satisfies the system (3) corresponding to this Lie algebra. Therefore, the symmetrized polynomial is the Casimir operator of $\mathfrak{g}$.

The main objective of this work is to show the correctness of this observation for algebras satisfying the constraint $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, i.e., being perfect. This apparent restriction is necessary, since for perfect Lie algebras the existence of complete sets of invariants formed by polynomials is ensured [8], while for non-perfect algebras we can even have transcendental invariants [8, 19, 22]. However, we will show that the Maurer–Cartan are also useful to find the commuting polynomials that constitute the rational invariant for Lie algebras with a codimension-1 commutator subalgebra. This also allows a method to find the Casimir operators of various Lie algebras with two invariants.

3. Inhomogeneous Lie algebras

Among all Lie algebras having a non-trivial Levi decomposition, inhomogeneous Lie algebras, i.e., semidirect products $\mathfrak{g} = \mathfrak{s}\overrightarrow{\oplus}_R(\dim R)L_1$ of semisimple and Abelian Lie algebras, have an important property concerning the structure of their Casimir invariants. As already told, it is known that these algebras admit polynomial invariants whenever the constraint $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ is satisfied [8]. It is natural to ask whether the Levi decomposition induces some kind of homogeneity of the Casimir operators with respect to the variables of the Levi subalgebra $\mathfrak{s}$ and the variables of the Abelian radical. This property, which holds exclusively for this class, is a consequence of the so-called missing label problem associated with these algebras [22] and the natural contraction related to it [23, 24].

Theorem 1. *Let $\mathfrak{g} = \mathfrak{s}\overrightarrow{\oplus}_R(\dim R)L_1$. Then any Casimir invariant is a homogeneous polynomial in the variables of the radical* $(\dim R)L_1$.

We also remark that the particular structure of the invariants of inhomogeneous Lie algebras is deeply connected with the expansion method considered by Rosen in [6] for unitary and pseudo-orthogonal algebras. Actually, the previous homogeneity with respect to the appropriate variables is one of the principal reasons for the possibility of recovering the Casimir operators of semisimple Lie algebras from those of an inhomogeneous contraction.

We now focus on inhomogeneous Lie algebras satisfying the constraint $\mathcal{N}(\mathfrak{g}) = 1$. For this case, various interesting structural properties emerge that will allow us to find an intrinsic construction of the Casimir operator.

Lemma 1. *Let* $\mathcal{L} = \{\omega_1, \ldots, \omega_{2r+1}\}$ *be a system of independent 1-forms such that* $\mathrm{d}\omega_i \in \bigwedge^2 \mathcal{L} - \{0\}$ *for* $i = 1 \cdots 2r+1$. *Let* $\omega = \sum_{i=1}^{2r+1} a_i\omega_i \in \mathcal{L}$ *be an element such that*

$$\left(\bigwedge^r \mathrm{d}\omega\right) \wedge \omega = \Phi(a_1, \ldots, a_{2r+1})\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_{2r+1} \neq 0,$$

where $\Phi(a_1, \ldots, a_{2r+1})$ *is a polynomial in the variables* a_i. *Then*

$$\bigwedge^r \mathrm{d}\omega = \frac{1}{r+1}\sum_{i=1}^{2r+1}(-1)^{i+1}\frac{\partial\Phi}{\partial a_i}\omega_1 \wedge \cdots \wedge \widehat{\omega_i} \wedge \cdots \wedge \omega_{2r+1}, \tag{13}$$

where $\widehat{\omega_i}$ *denotes omission of the element* ω_i.

Proof. From the properties of the wedge product it follows at once that $\Phi(a_1, \ldots, a_{2r+1})$ is a homogeneous polynomial of degree $r+1$ in the variables $a_1, \ldots, a_{2r+1}$. Define the $2r$-form

$$\theta = \sum_{i=1}^{2r+1}(-1)^{i+1}\frac{\partial\Phi}{\partial a_i}\omega_1 \wedge \cdots \wedge \widehat{\omega_i} \wedge \cdots \wedge \omega_{2r+1}. \tag{14}$$

By assumption, $\Phi \neq 0$, thus the 2-form θ is non-vanishing. Considering the 1-form $\omega = \sum_{i=1}^{2r+1} a_i\omega_i$ and taking the wedge product of θ and ω, we obtain that

$$\begin{aligned}\theta \wedge \omega &= \sum_{i=1}^{2r+1}(-1)^{i+1}\frac{\partial\Phi}{\partial a_i}\omega_1 \wedge \cdots \wedge \widehat{\omega_i} \wedge \cdots \wedge \omega_{2r+1} \wedge \left(\sum_{i=1}^{2r+1} a_i\omega_i\right)\\ &= \sum_{i=1}^{2r+1}(-1)^{i+1}a_i\frac{\partial\Phi}{\partial a_i}\omega_1 \wedge \cdots \wedge \widehat{\omega_i} \wedge \cdots \wedge \omega_{2r+1} \wedge \omega_i\\ &= \sum_{i=1}^{2r+1}(-1)^{2r+2}a_i\frac{\partial\Phi}{\partial a_i}\omega_1 \wedge \cdots \wedge \omega_i \wedge \cdots \wedge \omega_{2r+1}.\end{aligned} \tag{15}$$

By homogeneity of the polynomial Φ, the Euler identities imply the equality

$$\sum_{i=1}^{2r+1} a_i\frac{\partial\Phi}{\partial a_i} = (r+1)\Phi, \tag{16}$$

from which the assertion follows, since $\left((r+1)\bigwedge^r \mathrm{d}\omega - \theta\right) \wedge \omega = 0$ by equation (15). □

This result is formally very close to the approach by means of total differential equations as presented in (5). We will later see that actually the use of wedge products determines the Casimir operator, corresponding to the first integrals of (5). Moreover, in contrast to the general case, the constraint of having only one Casimir invariant imposes some restrictions on the possible dimension of a representation. The following results establish (sharp) bounds for the dimension of the radical in inhomogeneous Lie algebras with only one invariant.

Lemma 2. *Let* $\mathfrak{g} = \mathfrak{s}\overrightarrow{\oplus}_R nL_1$ *be an indecomposable Lie algebra. If* $\mathcal{N}(\mathfrak{g}) = 1$, *then* $\dim R = n \geqslant \operatorname{rank}\mathfrak{s} + 1$.

Proof. We consider the Levi subalgebra $\mathfrak{s}$ of $\mathfrak{g}$. It is known that, in addition to the Casimir operators of these algebras, we need

$$\tfrac{1}{2}(\dim\mathfrak{g} - \mathcal{N}(\mathfrak{g}) - \dim\mathfrak{h} - \mathcal{N}(\mathfrak{h})) + l' \tag{17}$$

additional operators (commonly called missing label operators) in order to label unambiguously the states of $\mathfrak{g}$ with respect to the subalgebra $\mathfrak{s}$ [22]. For this reduction, l' is clearly zero, thus we need

$$\tfrac{1}{2}(\dim\mathfrak{s} + n - 1 - \dim\mathfrak{s} - \mathrm{rank}\,\mathfrak{s}) = \tfrac{1}{2}(n - 1 - \mathrm{rank}\,\mathfrak{s}) \geqslant 0 \tag{18}$$

additional operators. The latter being a non-negative quantity, we conclude that $\dim R = n \geqslant 1 + \mathrm{rank}\,\mathfrak{s}$. □

We observe that the example presented at the beginning exactly satisfies the equality, showing that the lower bound is sharp. As to upper bounds, the result is a consequence of the specific structure of system (3) for these algebras:

Lemma 3. *Let* $\mathfrak{g} = \mathfrak{s}\overrightarrow{\oplus}_R nL_1$ *be an indecomposable Lie algebra. If* $\mathcal{N}(\mathfrak{g}) = 1$, *then* $\dim R = n \leqslant \dim\mathfrak{s} + 1$.

Proof. Consider a basis $\{X_1, \ldots, X_r, Y_1, \ldots, Y_n\}$ of $\mathfrak{g}$ such that $\{X_1, \ldots, X_r\}$ is a basis of the Levi part $\mathfrak{s}$ and $\{Y_1, \ldots, Y_n\}$ is a basis of the representation space R[1]. The invariant of $\mathfrak{g}$ is obtained from the system (3), expressed in matrix form

$$\begin{pmatrix} \widehat{X}_i F \\ \widehat{Y}_i F \end{pmatrix} = \begin{pmatrix} C_{ij}^k x_k & C_{ij}^k y_k \\ -C_{ij}^k y_k & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial F}{\partial x_i} \\ \frac{\partial F}{\partial y_i} \end{pmatrix} = 0. \tag{19}$$

In particular, the number $\mathcal{M}(\mathfrak{g})$ of invariants which depend only on the variables $\{y_1, \ldots, y_n\}$ is given by

$$\mathcal{M}(\mathfrak{g}) = \dim R - \mathrm{rank}\big(C_{ij}^k y_k\big), \tag{20}$$

where $B = \big(C_{ij}^k y_k\big)$ is an $(n \times r)$-matrix describing the action of the Levi part $\mathfrak{s}$ on the radical R [13]. In any case, the following inequality holds:

$$\mathrm{rank}\,\big(C_{ij}^k y_k\big) \leqslant \min\{n, r\}. \tag{21}$$

In view of this bound, there are only two possibilities:

(i) If $\mathcal{M}(\mathfrak{g}) = 0$, then $\dim R = n = \mathrm{rank}\big(C_{ij}^k y_k\big)$, which implies that $\dim R \leqslant \dim\mathfrak{s}$, as the condition $\dim R > \dim\mathfrak{g}$ would contradict inequality (21). Further, since equality of the dimensions of R and $\mathfrak{s}$ is forbidden by parity, we conclude that

$$\dim R \leqslant \dim\mathfrak{s} - 1. \tag{22}$$

(ii) If $\mathcal{M}(\mathfrak{g}) = 1$, then the Casimir operator of $\mathfrak{g}$ depends only on variables of the Abelian radical. In this case,

$$1 = \dim R - \mathrm{rank}\big(C_{ij}^k y_k\big), \tag{23}$$

and the only relevant case $n > r$ implies that $\mathrm{rank}\big(C_{ij}^k y_k\big) = n - 1 \leqslant r = \dim\mathfrak{s} < n$ and thus that

$$\dim R \leqslant \dim\mathfrak{s} + 1. \tag{24}$$

□

[1] Since the radical is Abelian, we can naturally identify it with the representation space R, where $n = \dim R$.

As a consequence of these lemmas, the representation R describing the semidirect product $\mathfrak{g} = \mathfrak{s}\overrightarrow{\oplus}_R nL_1$ with $\mathcal{N}(\mathfrak{g}) = 1$ satisfies rank $\mathfrak{s}+1 \leqslant \dim R \leqslant \dim\mathfrak{s}-1$ or $\dim R = \dim\mathfrak{s}+1$. The latter constitutes an isolated type of algebras, and it follows from the proof that their Casimir operator depends only on the variables of the radical [13]. The other possibility, $\dim R < \dim\mathfrak{s}$, presents additional features that are of relevance for the Casimir operator. Like before, suppose that $\{X_1,\ldots,X_r,Y_1,\ldots,Y_n\}$ is a basis of $\mathfrak{g}$ such that $\{X_1,\ldots,X_r\}$ is a basis of $\mathfrak{s}$ and $\{Y_1,\ldots,Y_n\}$ a basis of the radical nL_1. Denote the dual basis by $\{\omega_1,\ldots,\omega_r,\theta_1,\ldots,\theta_n\}$.

Lemma 4. *If there exists an element $\omega \in \mathfrak{g}^*$ such that*

$$\left(\bigwedge^{j_0(\mathfrak{g})} \mathrm{d}\omega\right) \wedge \omega = \Phi(a_1,\ldots,a_r,b_1,\ldots,b_n)\omega_1\wedge\cdots\wedge\omega_r\wedge\theta_1\wedge\cdots\wedge\theta_n \neq 0, \tag{25}$$

then $\Phi(a_i,b_j)$ is a homogeneous polynomial in both the variables $\{a_i\}$ and $\{b_j\}$.

Proof. Since $\mathcal{N}(\mathfrak{g}) = 1$, we obtain from (10) that

$$j_0(\mathfrak{g}) = \tfrac{1}{2}(n+r-1). \tag{26}$$

For the Levi subalgebra $\mathfrak{s}$ we have the equality $\dim\mathfrak{s} = r = \text{rank }\mathfrak{s} + 2j_0(\mathfrak{s})$, and insertion into equation (26) gives

$$j_0(g) = j_0(\mathfrak{s}) + \tfrac{1}{2}(\text{rank }\mathfrak{s} - 1 + n). \tag{27}$$

Let us consider a generic element $\omega\in\mathfrak{g}^*$:

$$\omega = \sum_{i=1}^{r} a_i\omega_i + \sum_{j=1}^{n} b_j\theta_j, \tag{28}$$

where $a_i, b_j \in \mathbb{R}$ are arbitrary constants. The exterior differential d is determined by

$$\mathrm{d}\omega = \sum_{i=1}^{r} a_i\,\mathrm{d}\omega_i + \sum_{j=1}^{n} b_j\,\mathrm{d}\theta_j. \tag{29}$$

Wedge products of the latter 2-form can be rewritten as

$$\bigwedge^{p}\mathrm{d}\omega = \bigwedge^{p}(\xi_1+\xi_2) = \sum_{k=1}^{p}\binom{p}{k}\bigwedge^{k}\xi_1\wedge\bigwedge^{p-k}\xi_2, \tag{30}$$

where $\xi_1 = \sum_{i=1}^{r} a_i\,\mathrm{d}\omega_i$ and $\xi_2 = \sum_{j=1}^{n} b_j\,\mathrm{d}\theta_j$. In particular, any summand of (30) is a wedge product of $k+p$ elements ω_i and $p-k$ elements θ_j. We say that $\bigwedge^p(\xi_1+\xi_2)$ has bi-degree $(k+p, p-k)$ in the $(\mathfrak{s}, R)$-variables. It follows at once from this that any term of $\bigwedge^p(\xi_1+\xi_2)$ has degree k in the variables a_i and degree $p-k$ in b_i's. Further, it is clear from (10) that $\bigwedge^p\xi_1 = 0$ for any $p > j_0(\mathfrak{s})$, thus

$$\bigwedge^{j_0(\mathfrak{g})}(\xi_1+\xi_2) = \sum_{k\geqslant 0}\binom{j_0(\mathfrak{g})}{k}\bigwedge^{j_0(\mathfrak{s})-k}\xi_1\wedge\bigwedge^{\frac{1}{2}(\text{rank }\mathfrak{s}-1+n)+k}\xi_2. \tag{31}$$

By assumption, $\bigwedge^{j_0(\mathfrak{g})}\mathrm{d}\omega\wedge\omega \neq 0$, thus applying the preceding lemma we obtain the decomposition:

$$\begin{aligned}\bigwedge^{j_0(\mathfrak{g})}(\xi_1+\xi_2) &= \frac{1}{j_0(\mathfrak{g})+1}\sum_{i=1}^{r}(-1)^{i+1}\frac{\partial\Phi}{\partial a_i}\omega_1\wedge\cdots\wedge\widehat{\omega_i}\wedge\cdots\wedge\omega_r\wedge\theta_1\wedge\cdots\wedge\theta_i\wedge\cdots\wedge\theta_n\\ &\quad+\frac{1}{j_0(\mathfrak{g})+1}\sum_{i=1}^{n}(-1)^{n+i+1}\frac{\partial\Phi}{\partial b_i}\omega_1\wedge\cdots\wedge\omega_i\wedge\cdots\wedge\omega_r\wedge\theta_1\wedge\cdots\wedge\widehat{\theta_i}\wedge\cdots\wedge\theta_n.\end{aligned} \tag{32}$$

The bi-degree of the terms in the latter product are either $(r-1,n)$ or $(r,n-1)$, since the bi-degree of Φ is (r,n). In view of equation (13), this decomposition is possible only if

$$\bigwedge^{j_0(\mathfrak{g})}(\xi_1+\xi_2) = \binom{j_0(\mathfrak{g})}{\frac{n-\text{rank }\mathfrak{s}-1}{2}} \bigwedge^{\frac{\dim\mathfrak{s}-n-1}{2}}\xi_1\wedge\bigwedge^{n}\xi_2 + \binom{j_0(\mathfrak{g})}{\frac{n-\text{rank }\mathfrak{s}-1}{2}} \bigwedge^{\frac{\dim\mathfrak{s}-n+1}{2}}\xi_1\wedge\bigwedge^{n-1}\xi_2, \tag{33}$$

because these are the only products having the required bi-degrees. Finally, considering the wedge product of (33) with ω, the following identities can be easily verified:

$$\bigwedge^{\frac{\dim\mathfrak{s}-n-1}{2}}\xi_1\wedge\bigwedge^{n}\xi_2\wedge\left(\sum_{i=1}^{r}a_i\omega_i+\sum_{j=1}^{n}b_j\theta_j\right) = \bigwedge^{\frac{\dim\mathfrak{s}-n-1}{2}}\xi_1\wedge\bigwedge^{n}\xi_2\wedge\left(\sum_{i=1}^{r}a_i\omega_i\right), \tag{34}$$

$$\bigwedge^{\frac{\dim\mathfrak{s}-n+1}{2}}\xi_1\wedge\bigwedge^{n-1}\xi_2\wedge\left(\sum_{i=1}^{r}a_i\omega_i+\sum_{j=1}^{n}b_j\theta_j\right) = \bigwedge^{\frac{\dim\mathfrak{s}-n+1}{2}}\xi_1\wedge\bigwedge^{n-1}\xi_2\wedge\left(\sum_{j=1}^{n}b_j\theta_j\right). \tag{35}$$

This shows that any term of Φ has degree n in the variables $\{b_j\}$ and degree $\frac{r-n+1}{2}$ in the variables a_i, and therefore that the polynomial Φ is homogeneous in these variables. □

This result is a sharpened version of theorem 1 and constitutes the essential step to recover the Casimir operator intrinsically.

4. Construction of the Casimir operator from the Maurer–Cartan equations

In this section we prove that for inhomogeneous Lie algebras possessing only one Casimir operator the latter can be constructed using only the Maurer–Cartan equations, based on the homogeneity properties previously seen. In [14] a result of similar nature was proposed, using a special extension of degree 1 of the algebra. We will use this result for a stronger statement, for which reason we briefly recall the result.

Proposition 1. *Let $\mathfrak{s}\overrightarrow{\oplus}_R nL_1$ be a perfect Lie algebra such that $\mathcal{N}(\mathfrak{s}\overrightarrow{\oplus}_R nL_1)=1$. Let C be the invariant of minimal degree. Then $\mathfrak{s}\overrightarrow{\oplus}_R nL_1$ admits an extension $\widehat{\mathfrak{g}}$ of degree 1 satisfying $\mathcal{N}(\widehat{\mathfrak{g}})=0$ and such that $|A(\mathfrak{g})|=C^2$ is the square power of C.*

Expressed in terms of the Maurer–Cartan equations $\{\mathrm{d}\varphi_1,\ldots,\mathrm{d}\varphi_{n+r+1}\}$ of the extension $\widehat{\mathfrak{g}}$, this result establishes the existence of an element $z^i\,\mathrm{d}\varphi_i\in\mathcal{L}(\widehat{\mathfrak{g}})$ such that

$$\bigwedge^{\frac{n+r+1}{2}}\mathrm{d}\omega=\left(\frac{n+r+1}{2}\right)!C(z_i)\varphi_1\wedge\cdots\wedge\varphi_{n+r+1}. \tag{36}$$

The reason for this remarkable relation between certain extensions and the Casimir operator of $\mathfrak{g}$ remained however unexplained in [14]. We will see that the Casimir operator of $\mathfrak{g}$, being completely determined by the Maurer–Cartan equations themselves, implies the existence of such an extension.

Theorem 2. *Let $\mathfrak{g}=\mathfrak{s}\overrightarrow{\oplus}_R nL_1$ be an indecomposable Lie algebra with $\mathcal{N}(\mathfrak{g})=1$. Then the Casimir operator C of minimal degree is intrinsically determined by the Maurer–Cartan equations of $\mathfrak{g}$.*

Proof. We consider a basis $\mathcal{B} = \{X_1, \ldots, X_r, Y_1, \ldots, Y_n\}$, where $\{X_1, \ldots, X_r\}$ is a basis of the Levi part $\mathfrak{s}$ and $\{Y_1, \ldots, Y_n\}$ is a basis of the representation space R. The structure tensor over $\mathcal{B}$ is given by

$$[X_i, X_j] = C_{ij}^k X_k, \qquad [X_i, Y_j] = D_{ij}^k Y_k, \qquad [Y_i, Y_j] = 0.$$

Considering the dual basis $\{\omega_1, \ldots, \omega_r, \theta_1, \ldots, \theta_n\}$ to $\mathcal{B}$, the Maurer–Cartan equations of $\mathfrak{g}$ are easily seen to be

$$\begin{aligned} \mathrm{d}\omega_i &= C_{jk}^i \omega_j \wedge \omega_k, & i &= 1, \ldots, r, \\ \mathrm{d}\theta_j &= D_{ik}^j \omega_i \wedge \theta_k, & j &= 1, \ldots, n. \end{aligned} \tag{37}$$

By assumption, $\mathcal{N}(\mathfrak{g}) = 1$, thus by formula (10) the condition $j_0(\mathfrak{g}) = \frac{n+r-1}{2}$ is satisfied. Let $\widehat{\mathfrak{g}}$ denote the extension of $\mathfrak{g}$ containing it as a codimension-1 ideal. The Maurer–Cartan equations of $\widehat{\mathfrak{g}}$ are given, over the basis $\{\widetilde{\omega}_1, \ldots, \widetilde{\omega}_r, \widetilde{\theta}_1, \ldots, \widetilde{\theta}_n, \xi\}$, by

$$\begin{aligned} \mathrm{d}\widetilde{\omega}_i &= \mathrm{d}\omega_i, & 1 &\leqslant i \leqslant r \\ \mathrm{d}\widetilde{\theta}_j &= \mathrm{d}\theta_j + \theta_j \wedge \xi, & 1 &\leqslant j \leqslant n \\ \mathrm{d}\xi &= 0. \end{aligned} \tag{38}$$

Following proposition 1, $\widehat{\mathfrak{g}}$ satisfies $\mathcal{N}(\widehat{\mathfrak{g}}) = 0$, which implies the condition $j_0(\widehat{\mathfrak{g}}) = \frac{n+r+1}{2} = j_0(\mathfrak{g}) + 1$. We can therefore find an element $\Omega = \sum_{i=1}^r a_i \,\mathrm{d}\widetilde{\omega}_i + \sum_{j=1}^n b_j \,\mathrm{d}\widetilde{\theta}_j$ such that

$$\begin{aligned} \bigwedge^{\frac{n+r+1}{2}} \Omega &= \frac{n+r-1}{2} C(a_i, b_j) \widetilde{\omega}_1 \wedge \cdots \wedge \widetilde{\omega}_r \wedge \widetilde{\theta}_1 \wedge \cdots \wedge \widetilde{\theta}_n \wedge \xi \\ &= \frac{n+r-1}{2} C(a_i, b_j) \omega_1 \wedge \cdots \wedge \omega_r \wedge \theta_1 \wedge \cdots \wedge \theta_n \wedge \xi, \end{aligned} \tag{39}$$

where $C(x_i, y_j)$ is the Casimir invariant of $\mathfrak{g}$, after replacing a_i by x_i and b_j by y_j. We decompose the 2-form Ω as follows:

$$\begin{aligned} \Omega &= d\left(\sum_{i=1}^r a_i \widetilde{\omega}_i + \sum_{j=1}^n b_j \widetilde{\theta}_j + \xi\right) = d\left(\sum_{i=1}^r a_i \omega_i + \sum_{j=1}^n b_j \widetilde{\theta}_j\right) \\ &= d\left(\sum_{i=1}^r a_i \omega_i + \sum_{j=1}^n b_j \theta_j\right) + \left(\sum_{j=1}^n b_j \theta_j\right) \wedge \xi = \mathrm{d}\omega + \left(\sum_{j=1}^n b_j \theta_j\right) \wedge \xi, \end{aligned} \tag{40}$$

where $\omega = \sum_{i=1}^r a_i \omega_i + \sum_{j=1}^n b_j \theta_j$. It follows by induction on $k \geqslant 1$ that wedge products of Ω can be rewritten in the following way:

$$\bigwedge^k \Omega = \bigwedge^k \mathrm{d}\omega + k \left(\bigwedge^{k-1} \mathrm{d}\omega\right) \wedge \left(\sum_{j=1}^n b_j \theta_j\right) \wedge \xi. \tag{41}$$

In particular, for $k = j_0(\widehat{\mathfrak{g}})$ the product reduces to

$$\bigwedge^{j_0(\widehat{\mathfrak{g}})} \Omega = \frac{n+r+1}{2} \left(\bigwedge^{j_0(\mathfrak{g})} \mathrm{d}\omega\right) \wedge \left(\sum_{j=1}^n b_j \theta_j\right) \wedge \xi. \tag{42}$$

Combining the latter with equation (39), we obtain the expression:

$$\begin{aligned} &\frac{n+r+1}{2} \left(\bigwedge^{j_0(\mathfrak{g})} \mathrm{d}\omega\right) \wedge \left(\sum_{j=1}^n b_j \theta_j\right) \wedge \xi \\ &\quad = \left(\frac{n+r+1}{2}\right)! C(a_i, b_j) \omega_1 \wedge \cdots \wedge \omega_r \wedge \theta_1 \wedge \cdots \wedge \theta_n \wedge \xi. \end{aligned} \tag{43}$$

Now consider the product

$$\left(\bigwedge^{j_0(\mathfrak{g})} \mathrm{d}\omega\right) \wedge \omega = \Phi(a_i, b_j)\omega_1 \wedge \cdots \wedge \omega_r \wedge \theta_1 \wedge \cdots \wedge \theta_n, \tag{44}$$

where Φ is a homogeneous polynomial of degree $\frac{n+r+1}{2}$.[2] The decomposition of lemma 1 implies the following identity:

$$\begin{aligned}\bigwedge^{j_0(\mathfrak{g})} \mathrm{d}\omega = \sum_{i=1}^{r}(-1)^{i+1}\frac{\partial\Phi}{\partial a_i}\omega_1 \wedge \cdots \wedge \widehat{\omega_i} \wedge \cdots \wedge \omega_r \wedge \theta_1 \wedge \cdots \wedge \theta_n \\ + \sum_{j=1}^{n}(-1)^{n+j+1}\frac{\partial\Phi}{\partial b_j}\omega_1 \wedge \cdots \wedge \omega_r \wedge \theta_1 \wedge \cdots \wedge \widehat{\theta_j} \wedge \cdots \wedge \theta_n.\end{aligned} \tag{45}$$

Taking now the wedge product

$$\left(\bigwedge^{j_0(\mathfrak{g})} \mathrm{d}\omega\right) \wedge \left(\sum_{j=1}^{n} b_j\theta_j\right) = \sum_{j=1}^{n} b_j\frac{\partial\Phi}{\partial b_j}\omega_1 \wedge \cdots \wedge \omega_r \wedge \theta_1 \wedge \cdots \wedge \theta_j \wedge \cdots \wedge \theta_n, \tag{46}$$

and comparing it with (43) we conclude the following relation between Φ and the Casimir operator:

$$\frac{n+r+1}{2}\sum_{j=1}^{n} b_j\frac{\partial\Phi}{\partial b_j} = \left(\frac{n+r+1}{2}\right)!C(a_i, b_j). \tag{47}$$

Observe that if $n = r + 1$, by lemma 2 the following constraint is satisfied:

$$\frac{\partial C(a_i, b_j)}{\partial a_k} = 0, \qquad k = 1, \ldots, r, \tag{48}$$

and by homogeneity we conclude that Φ is a multiple of the Casimir operator of $\mathfrak{g}$. This proves the result for the case of maximal dimension of the representation R. If $n < r$, lemma 3 implies that Φ is homogeneous in the variables b_j (and therefore also in the variables a_i). Applying the Euler theorem, we obtain the identity

$$\sum_{j=1}^{n} b_j\frac{\partial\Phi}{\partial b_j} = (\deg_b \Phi)\Phi = n\Phi. \tag{49}$$

Inserting the latter expression into equation (43) gives the relation

$$n\frac{n+r+1}{2}\Phi = \left(\frac{n+r+1}{2}\right)!C(a_i, b_j), \tag{50}$$

which proves that Φ is a Casimir operator of $\mathfrak{g}$ after replacement of a_i by x_i and b_j by y_j. In particular, for any possible dimension of R we obtain that $\Phi \neq 0$, and therefore the Casimir operator of $\mathfrak{g}$ is uniquely determined by the wedge product (44) of the Maurer–Cartan equations. □

This result provides us with an algorithmic procedure to compute the Casimir operator starting from an arbitrary basis of the algebra. Moreover, equation (44) shows that performing wedge products of the defining equations of $\mathfrak{g}$ and extending it to an algebra satisfying the condition $\mathcal{N}(\widehat{\mathfrak{g}}) = 0$ leads to the same result. We remark that the key point is the homogeneity of the Casimir operator with respect to the variables of the Levi subalgebra and the radical. Incidentally, this method implies the existence of a supplementary geometrical property.

[2] At this point it is not excluded that this expression vanishes, i.e., that Φ is a zero polynomial.

Contact forms constitute, in some sense, an analogous concept to symplectic forms for odd dimensional manifolds. Although generally weaker than the symplectic frame of Hamiltonian mechanics, contact structures appear naturally in many physical problems, such as generalizations of magnetic monopoles [25], irreversible thermodynamical systems [26] or geometric formulations of gravity coupled with Yang–Mills fields [27]. We recall that a linear contact form on a Lie algebra $\mathfrak{g}$ of dimension $2m+1$ is an element $\omega \in \mathfrak{g}^*$ such that $\omega \wedge \left(\bigwedge^m \mathrm{d}\omega\right) \neq 0$. In particular, formula (10) implies that $\mathcal{N}(\mathfrak{g}) = 1$, although the converse is not necessarily true.

Proposition 2. *If* $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] = \mathfrak{s} \overrightarrow{\oplus}_R nL_1$ *is indecomposable and satisfies* $\mathcal{N}(\mathfrak{g}) = 1$, *then* $\mathfrak{g}$ *is endowed with a linear contact form.*

The proof follows easily from the previous argumentation. Again, two cases must be considered according to the dimension of the Abelian radical. If $n = r+1$, equation (47) shows that the 1-form defined by $\xi = \sum_{j=1}^{n} b_j \theta_j$ is a contact form. In this case, equations (45) and (46) imply the relation

$$\bigwedge^{\frac{n+r-1}{2}} \mathrm{d}\xi \wedge \xi = \left(\frac{n+r}{2}\right)! C(b_j) \omega_1 \wedge \cdots \wedge \omega_r \wedge \theta_1 \wedge \cdots \wedge \theta_n. \tag{51}$$

On the other hand, if $n < r$, equation (49) establishes that $\sum_{j=1}^{n} b_j \frac{\partial \Phi}{\partial b_j} = n\Phi$. Applying the Euler theorem to Φ we obtain

$$\sum_{i=1}^{r} a_i \frac{\partial \Phi}{\partial a_i} + \sum_{j=1}^{n} b_j \frac{\partial \Phi}{\partial b_j} = \sum_{i=1}^{r} a_i \frac{\partial \Phi}{\partial a_i} + n\Phi = \frac{n+r+1}{2}\Phi, \tag{52}$$

thus

$$\sum_{i=1}^{r} a_i \frac{\partial \Phi}{\partial a_i} = \frac{r+1-n}{2}\Phi \neq 0, \tag{53}$$

and from equation (44) we conclude that ω defines a linear contact form on $\mathfrak{g}$.

Therefore the contact form is deeply related to the homogeneity properties of the Casimir operator, but also to those of the extension $\widehat{\mathfrak{g}}$ used in [14], which is derived from it. This naturally explains why the determinant method developed there holds. In particular, the contact form implies that these algebras contract onto the Heisenberg algebra of the same dimension [13].

4.1. The Lie algebras $\mathfrak{sa}(N, \mathbb{R})$

The physically most important non-simple Lie algebra with only one Casimir operator is the special affine algebra $\mathfrak{sa}(n, \mathbb{R}) = \mathfrak{sl}(n, \mathbb{R}) \overrightarrow{\oplus}_R nL_1$ appearing in quantum gauge theories of gravity [2, 11]. The invariant has been used both in the classification of particles and in establishing a wave equation [3]. Various works have been devoted to the problem of finding explicit expressions of the Casimir operator of this algebra, either from the analytical point of view [2, 28], with algebraic procedures [14], or more recently with the tensor approach of enveloping algebras [16]. The advantage of the Maurer–Cartan equations is the possibility of computing the Casimir operator starting from an arbitrary basis. Taking for example the boson realization of $\mathfrak{sa}(N, \mathbb{R})$ given by

$$X_\mu = b_\mu^+ b_\mu^- - b_{\mu+1}^+ b_{\mu+1}^-, \qquad X_{\mu\nu} = b_\mu^+ b_\nu^-, \qquad Y_\nu = b_\nu^+, \tag{54}$$

where $\left[b_i^+, b_j^+\right] = \left[b_i^-, b_j^-\right] = 0, \left[b_i^-, b_j^+\right] = \delta_i^j$, the corresponding Maurer–Cartan equations are given by

$$\begin{aligned}
\mathrm{d}\omega_\mu &= \sum_{\rho=1}^{\mu} \sum_{\nu=\mu+1}^{N} \omega_{\rho\nu} \wedge \omega_{\nu\rho}, \qquad 1 \leqslant \mu \leqslant N-1 \\
\mathrm{d}\omega_{\nu\rho} &= \sum_{\mu=1}^{N-1} \left(\delta_\nu^\mu + \delta_\rho^{\mu+1} - \delta_\rho^\nu - \delta_\nu^{\mu+1}\right) \omega_\mu \wedge \omega_{\nu\rho} + \sum_{\sigma=1}^{N} \omega_{\nu\sigma} \wedge \omega_{\sigma\rho}, \\
\mathrm{d}\theta_\rho &= \delta_\mu^\rho \omega_\mu \wedge \theta_\rho - \delta_{\mu+1}^\rho \omega_\mu \wedge \theta_\rho + \delta_\mu^\nu \omega_{\rho\nu} \wedge \theta_\mu.
\end{aligned}$$

Let $\alpha^\mu, \beta^{\mu\nu}, \gamma^\rho \in \mathbb{R}$ be constants and define the 1-form

$$\psi = \alpha^\mu \omega_\mu + \beta^{\nu\rho} \omega_{\nu\rho} + \gamma^\sigma \theta_\sigma.$$

Then,

$$\psi \wedge \left(\bigwedge^{\frac{1}{2}(N^2+N-2)} \mathrm{d}\psi \right) = \left(\frac{N^2+N}{2}\right)! C\left(\alpha^\mu, \beta^{\nu\rho}, \gamma^\sigma\right) \left(\bigwedge_{\mu=1}^{N-1} \omega_\mu \right) \wedge \left(\bigwedge_{\nu<\rho} \omega_{\nu\rho} \right) \wedge \left(\bigwedge_{\sigma=1}^{N} \theta_\sigma \right). \tag{55}$$

Taking the polynomial $C(\alpha^\mu, \beta^{\nu\rho}, \gamma^\sigma)$ and replacing the variables

$$\alpha^\mu \longmapsto x_\mu, \qquad \beta^{\nu\rho} \longmapsto x_{\nu\rho}, \qquad \gamma^\sigma \longmapsto p_\sigma,$$

the symmetrization of $C(x_\mu, x_{\nu\rho}, p_\sigma)$ provides the Casimir operator of $\mathfrak{sa}(N, \mathbb{R})$. So, for example, for the lowest values of $N = 2, 3, 4$, the previous function $C(x_\mu, x_{\nu\rho}, p_\sigma)$ has 3, 58 and 8196 terms, coinciding with the result of [28].

5. Algebras with a rational invariant

The method, as presented, seems to be valid only for Lie algebras with one classical Casimir operator. However, since rational invariants of Lie algebras are the quotient of commuting polynomials, it can be asked whether for the case of one rational invariant, the commuting polynomials can be obtained from the corresponding Maurer–Cartan equations[3]. This turns out to be the case, as we shall show with some examples.

5.1. The extended Poincaré algebra

The Weyl group $W(1, 3)$, i.e., the Poincaré group $I\mathfrak{so}(1, 3)$ enlarged with a spacetime dilation operator D, appears naturally when studying the symmetries of a spinning particle which satisfies the Dirac equation when quantized [29], constitutes an important example of a Lie algebra exhibiting only one rational invariant. Using the standard basis $E_{\mu\nu} = -E_{\nu\mu}$, P_μ and metric $g = \mathrm{diag}(1, 1, 1, -1)$, the brackets are given by

$$\begin{aligned}
&[E_{\mu\nu}, E_{\lambda\sigma}] = g_{\mu\lambda} E_{\nu\sigma} + g_{\mu\sigma} E_{\lambda\nu} - g_{\nu\lambda} E_{\mu\sigma} - g_{\nu\sigma} E_{\lambda\mu}, \\
&[E_{\mu\nu}, P_\rho] = g_{\mu\rho} P_\nu - g_{\nu\rho} P_\mu, \qquad [D, P_\rho] = -P_\rho.
\end{aligned} \tag{56}$$

Denoting by $\{\omega_{\mu\nu}, \theta_\mu, \xi\}$ the dual basis, the Maurer–Cartan equations are simply

$$\mathrm{d}\omega_{\nu\sigma} = g_{\mu\lambda} \omega_{\mu\nu} \wedge \omega_{\lambda\sigma}, \qquad \mathrm{d}\theta_\nu = g_{\mu\rho} \omega_{\mu\nu} \wedge \theta_\rho - \xi \wedge \theta_\nu, \qquad \mathrm{d}\xi = 0. \tag{57}$$

[3] Observe that in this case the condition $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ is no more satisfied [8].

A generic 1-form $\psi = a^{\mu\nu}\omega_{mu\nu} + \beta^{\rho}\theta_{\rho} + \xi$ has coboundary operator $\mathrm{d}\psi = a^{\mu\nu}\,\mathrm{d}\omega_{mu\nu} + \beta^{\rho}\,\mathrm{d}\theta_{\rho}$. Since $\mathcal{N}(W(1,3)) = 1$, we obtain

$$\left(\bigwedge^{5} \mathrm{d}\psi\right) \wedge \psi = C_1(a^{\mu\nu}, \beta^{\rho})C_2(a^{\mu\nu}, \beta^{\rho})\omega_{12} \wedge \cdots \wedge \omega_{34} \wedge \theta_1 \wedge \cdots \wedge \theta_4, \tag{58}$$

where, after the corresponding replacements $\{a^{\mu\nu} \mapsto e_{\mu\nu}, b^{\rho} \mapsto p_{\rho}\}$, the polynomials are given by $C_1 = g^{\mu\mu} p_{\mu}^2$ and

$$\begin{aligned} C_2 = -2 \sum_{\mu<\nu<\rho} & g_{\mu\mu}g_{\nu\nu}g_{\rho\rho}(\epsilon_{\mu\nu\rho}p_{\mu}p_{\nu}e_{\mu\rho}e_{\nu\rho} + \epsilon_{\mu\rho\nu}p_{\mu}p_{\rho}e_{\mu\nu}e_{\nu\rho} + \epsilon_{\nu\rho\mu}p_{\nu}p_{\rho}e_{\nu\rho}e_{\mu\rho}) \\ & + \sum_{\mu<\nu} g_{\mu\mu}g_{\nu\nu}e_{\mu\nu}^2 \left(\sum_{\rho\neq\mu,\nu} g_{\rho\rho}p_{\rho}^2\right). \end{aligned} \tag{59}$$

These are the well-known Casimir operators of the Poincaré algebra the quotient C_2/C_1 of which provides the invariant of $W(1,3)$. Therefore, the Maurer–Cartan equations provide the commuting polynomials appearing in the rational invariant.

5.2. The optical Lie algebra

The optical Lie algebra Opt(1, 2) corresponds to the subalgebra of the anti De Sitter algebra $\mathfrak{so}(2,3)$ that leaves a lightlike vector invariant in Minkowski spacetime [30]. The algebra Opt(1, 2) is defined by the non-vanishing brackets

$$\begin{array}{llll} [K_1, K_2] = -K_3, & [K_1, K_3] = -K_2, & [K_2, K_3] = K_1, & [K_1, M] = -\frac{1}{2}M, \\ [K_1, Q] = \frac{1}{2}Q, & [K_2, M] = \frac{1}{2}Q, & [K_2, Q] = \frac{1}{2}M, & [K_3, M] = -\frac{1}{2}Q, \\ [K_3, Q] = \frac{1}{2}M, & [W, M] = \frac{1}{2}M, & [W, Q] = \frac{1}{2}Q, & [W, N] = N. \end{array} \tag{60}$$

over the basis $\{K_i, W, M, Q, N\}$. This algebra has only one invariant, which is a rational function [30]. Again, the Maurer–Cartan equations provide information. Denoting the dual basis by $\{\omega_1, \omega_2, \omega_3, \theta_1, \theta_2, \theta_3, \theta_4\}$ and considering a linear combination $\zeta = k_i\omega_i + w\theta_1 + m\theta_2 + q\theta_3 + n\theta_4$, the result of the wedge product

$$\bigwedge^{3} \mathrm{d}\zeta \wedge \zeta = 3n(q^2(k_2 + k_3) + m^2(k_3 - k_2) - 2k_1mq)\omega_1 \wedge \cdots \omega_3 \wedge \theta_1 \wedge \cdots \wedge \theta_4$$

shows that Opt(1, 2) is endowed with a contact form. Although the polynomial $C = 3n(q^2(k_2 + k_3) + m^2(k_3 - k_2) - 2k_1mq)$ in not an invariant of Opt(1, 2), it can be easily verified that any term is a semi-invariant, from which the rational invariant is easily deduced as the quotient of the terms $P = (q^2(k_2 + k_3) + m^2(k_3 - k_2) - 2k_1mq)/n$.

5.3. The two-photon algebra

The two-photon algebra $\mathfrak{h}_6$, isomorphic to the Schrödinger algebra in (1 + 1)-dimension, has been used, among other applications, to construct infinite classes of *N*-particle Hamiltonian systems [31, 32]. Over the basis $\{N, A_+, A_-, B_+, B_-, M\}$ the brackets are given by

$$\begin{array}{lll} [N, B_{\pm}] = \pm 2B_{\pm}, & [N, A_{\pm}] = \pm A_{\pm}, & [B_+, B_-] = -4N - 2M, \\ [B_+, A_-] = -2A_+, & [B_-, A_+] = 2A_-, & [A_+, A_-] = -M. \end{array}$$

This algebra clearly has two Casimir operators, one being the central charge M. To compute them using the Maurer–Cartan equations, we consider the extension of $\mathfrak{h}_6$ by an element Y acting on the two-photon algebra as follows:

$$[Y, N] = -M, \qquad [Y, A_{\pm}] = A_{\pm}, \qquad [Y, M] = 2M.$$

The seven-dimensional algebra $\mathfrak{g}$ satisfies $\mathcal{N}(\mathfrak{g}) = 1$. Let $\{\vartheta, \omega_1, \omega_2, \theta_1, \theta_2, \sigma, \chi\}$ be the dual basis to $\{N, A_+, A_-, B_+, B_-, M, Y\}$ and take the structure equations

$$\begin{aligned}
&\mathrm{d}\vartheta = -4\theta_1 \wedge \theta_2, \qquad \mathrm{d}\theta_1 = 2\vartheta \wedge \theta_1, \qquad \mathrm{d}\theta_2 = -2\vartheta \wedge \theta_2, \qquad \mathrm{d}\chi = 0,\\
&\mathrm{d}\omega_1 = \vartheta \wedge \omega_1 + 2\omega_2 \wedge \theta_1 + \omega_1 \wedge \chi,\\
&\mathrm{d}\omega_2 = -\vartheta \wedge \omega_2 - 2\omega_1 \wedge \theta_2 + \omega_2 \wedge \chi,\\
&\mathrm{d}\sigma = -2\theta_1 \wedge \theta_2 - \omega_1 \wedge \omega_2 - \vartheta \wedge \chi + 2\sigma \wedge \chi.
\end{aligned}$$

An arbitrary linear combination $\alpha = a_1\omega_1 + a_2\omega_2 + a_3\vartheta + a_4\theta_1 + a_5\theta_2 + a_6\sigma + a_7\chi$ gives rise to the wedge product

$$\begin{aligned}
\alpha \wedge \mathrm{d}\alpha \wedge \mathrm{d}\alpha \wedge \mathrm{d}\alpha = {}& 12a_6\left({a_6}^3 - 4a_6\,(a_4a_5 + a_1a_2) + a_3a_6(a_3 + a_6) + a_1^2a_5 + a_4a_2^2 - 2a_1a_2a_3\right)\\
&\times \omega_1 \wedge \omega_2 \wedge \vartheta \wedge \theta_1 \wedge \theta_2 \wedge \sigma \wedge \chi.
\end{aligned}$$

With the replacements

$$a_1 \mapsto a_+, \quad a_2 \mapsto a_-, \quad a_3 \mapsto n, \quad a_4 \mapsto b_+, \quad a_5 \mapsto b_-, \quad a_6 \mapsto m,$$

we obtain the polynomial

$$P = 12m\left(m^3 - 4m\,(b_+b_- + a_+a_-) + mn(n + m) + a_+^2b_- + b_+a_-^2 - 2a_+a_-n\right). \tag{61}$$

P is the Casimir operator of the extended algebra $\mathfrak{g}$, and it is straightforward to verify that $C_1 = m$ and $C_2 = -4m(b_+b_- + a_+a_-) + mn(n+m) + a_+^2b_- + b_+a_-^2 - 2a_+a_-n$ are the invariants of the subalgebra $\mathfrak{h}_6$.

Although in these examples the Maurer–Cartan equations do not provide the rational invariant itself, they give the commuting polynomials that intervene in it[4]. This means that the invariant has the form $C = C_1^aC_2^{-b}$, where only the values of a, b have to be checked. An interesting observation concerning these and other examples is that the degree of the product of these polynomials is always the integer part of half the dimension of the algebra plus one. Analyzing with the same method all Lie algebras of odd dimension $n \leqslant 9$ with one rational invariant and non-trivial Levi decomposition [9, 33], we always found the same rule for the degrees of C_1 and C_2. This enables us to suggest the following general property for Lie algebras:

Lemma 5. *Let* $\mathfrak{g} = \mathfrak{s}\overrightarrow{\oplus}_R\mathfrak{r}$ *be a Lie algebra satisfying the constraint* $\dim(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]) \leqslant 1$ *and* $\mathcal{N}(\mathfrak{g}) = 1$. *Then the rational invariant* $C = C_1^aC_2^{-b}$ *is such that*

$$\deg(C_1C_2) \leqslant \tfrac{1}{2}(\dim\mathfrak{g} + 1). \tag{62}$$

The previous examples correspond to the case $\dim(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]) = 1$, where $\mathfrak{g}' = [\mathfrak{g},\mathfrak{g}]$ is a codimension-1 subalgebra that satisfies $\mathfrak{g}' = [\mathfrak{g}',\mathfrak{g}']$, thus admits Casimir operators as invariants. Since $\mathcal{N}(\mathfrak{g}) = 1$, this means that $\mathcal{N}(\mathfrak{g}') = 2$, and therefore the polynomials C_1 and C_2 found for the rational invariant of $\mathfrak{g}$ can be taken as the Casimir operators of $\mathfrak{g}'$.

6. Final remarks

It has been proved that for Lie algebras $\mathfrak{g} = \mathfrak{s}\overrightarrow{\oplus}_R nL_1$ with one Casimir operator, the latter can be obtained by means of wedge products of the Maurer–Cartan equations. This constitutes a generalization, to non-simple Lie algebras, of the well-known fact that for the simple algebra $\mathfrak{su}(2)$ the quadratic Casimir operator arises from the structure equations. This fact shows moreover that for this class of algebras the invariant inherits a clear geometrical meaning as the function that appears in the volume form determined by the structure equations. The

[4] More specifically, they provide the Casimir operators of a codimension-1 subalgebra that is perfect.

proof is based on the homogeneity properties of the Casimir operators of inhomogeneous Lie algebras with respect to the variables of the Levi part and the representation describing the semidirect product. In particular, this property allows us to obtain sharp bounds for the dimension of the representation space R, whenever the constraint $\mathcal{N}(\mathfrak{g}) = 1$ is satisfied. This in principle enables us to classify inhomogeneous algebras with only one invariant, given a fixed Levi subalgebra. Further, it has been pointed out that the procedure based on differential forms can be extended for Lie algebras $\mathfrak{g}$ with only one rational invariant and a codimension-1 commutator subalgebra. This enables us further to reconstruct the Casimir operators of the latter as the terms appearing in the volume form associated with $\mathfrak{g}$, thus providing a geometrical method to compute the Casimir operators of Lie algebras with two invariants.

The procedure using differential forms is clearly limited to Lie algebras having rational invariants. Since the function appearing in the volume form obtained from the Maurer–Cartan equations is always polynomial, Lie algebras having transcendental invariants, like most solvable Lie algebras, are excluded. Whether introducing additional constraints to the ansatz of forms can provide a method to cover solvable algebras with a non-rational invariant is still an open question. Nowadays, the best possible known approach for the solvable case, using forms, is that of the moving frame method developed in [15, 19].

Finally, the differential-geometric derivation of the Casimir operator developed here serves to clarify a result concerning the degrees of a Casimir operator. In [11], it was claimed that for an n-dimensional Lie algebra $\mathfrak{g}$ with one invariant, the corresponding Casimir operator has either degree 1 or $\frac{n+1}{2}$. We give a counterexample to this claim, showing that the degree can be actually different. Consider the Lie algebra $L_{7,2}$ determined by the Maurer–Cartan equations

$$\begin{aligned}
&\mathrm{d}\omega_1 = \omega_2 \wedge \omega_3, \qquad \mathrm{d}\omega_2 = -\omega_1 \wedge \omega_3, \qquad \mathrm{d}\omega_3 = \omega_1 \wedge \omega_2,\\
&\mathrm{d}\omega_4 = -\tfrac{1}{2}(\omega_1 \wedge \omega_7 - \omega_2 \wedge \omega_5 + \omega_3 \wedge \omega_6), \qquad \mathrm{d}\omega_5 = -\tfrac{1}{2}(\omega_1 \wedge \omega_6 - \omega_2 \wedge \omega_4 + \omega_3 \wedge \omega_5),\\
&\mathrm{d}\omega_6 = \tfrac{1}{2}(\omega_1 \wedge \omega_5 - \omega_2 \wedge \omega_7 + \omega_3 \wedge \omega_4), \qquad \mathrm{d}\omega_7 = -\tfrac{1}{2}(\omega_1 \wedge \omega_4 + \omega_2 \wedge \omega_6 - \omega_3 \wedge \omega_5).
\end{aligned}$$

Since the algebra is indecomposable, it has no invariant of order one. Following theorem 1 of [11], the Casimir operator should be of order four. However, taking a generic element $\omega = a_i\omega_i$ and computing the corresponding wedge product $\omega \wedge \left(\bigwedge^3 \mathrm{d}\omega\right)$ we obtain the expression

$$\omega \wedge \left(\bigwedge^3 \mathrm{d}\omega\right) = \left(a_4^2 + a_5^2 + a_6^2 + a_7^2\right)^2 \omega_1 \wedge \cdots \wedge \omega_7,$$

and the function $\left(a_4^2 + a_5^2 + a_6^2 + a_7^2\right)^2$ is a square. It can be easily verified that $x_4^2 + x_5^2 + x_6^2 + x_7^2$ is a quadratic Casimir invariant of the algebra. The reason for the failure of the statement in [11] lies in the fact that the independence of the invariant on the variables of the Levi part was not explicitly considered in the proof. In this sense, the result of [11] should be reformulated as

Proposition 3. *If* $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ *has only one invariant C, then the order of C is either 1 or a divisor of* $\frac{1}{2}(\dim \mathfrak{g} + 1)$.

Acknowledgments

During the preparation of this work, the author was financially supported by the research project MTM2006-09152 of the M E C and the project and CCG07-UCM/ESP-2922 of the U C M-C A M.

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